

## A BERNSTEIN TYPE OF INEQUALITY FOR EIGENFUNCTIONS

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### Abstract

We proved a Bernstein type of inequality for eigenfunctions on Riemann Surfaces. This result improves similar results by H. Donnelly and C. Fefferman in two dimensions.

### 1. Introduction

The following Bernstein inequality is well known (cf. [8]). Let  $\phi(\theta)$  be a trigonometric polynomial of degree  $\leq n$ . Then,

$$(1.1) \quad \max_{0 \leq \theta \leq 2\pi} \left| \frac{\partial \phi}{\partial \theta} \right| \leq n \max_{0 \leq \theta \leq 2\pi} |\phi|.$$

A principal theme in [1], [2] is that an eigenfunction with eigenvalue  $\lambda$  on a closed manifold behaves like a polynomial of degree  $\sqrt{\lambda}$ . H. Donnelly and C. Fefferman proved in [1]

**Theorem 1.1.** *Let  $M$  be a smooth closed Riemannian manifold of dimension  $n$ , and  $u$  be an eigenfunction with eigenvalue  $\lambda$ . Then,*

$$(1.2) \quad \max_{B_r(x)} |\nabla u| \leq \frac{c_2 \lambda^{\frac{n+2}{4}}}{r} \max_{B_r(x)} |u|.$$

They also conjectured

**Conjecture 1.2.**

$$(1.3) \quad \max_{B_r(x)} |\nabla u| \leq \frac{c_4 \sqrt{\lambda}}{r} \max_{B_r(x)} |u|.$$

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The vanishing order estimates are simply corollaries of this inequality (cf. [1] and [9] for vanishing order estimates). Therefore, the Bernstein inequality reveals even deeper structure of eigenfunctions. Bernstein inequalities are also known to be associated to the construction of amplification devices.

The main results of this paper are the following:

**Theorem 1.3.** *Let  $M$  be a smooth closed Riemannian manifold of dimension two, and  $u$  be an eigenfunction with eigenvalue  $\lambda \geq 1$ . Then, for  $r \leq c_5 \lambda^{-1/4}$*

$$(1.4) \quad \max_{B_r(x)} |\nabla u| \leq \frac{c_7 \sqrt{\lambda}}{r} \max_{B_r(x)} |u|.$$

**Corollary 1.4.** *Under the same condition as above, for  $r \leq c_8 \lambda^{-1/4}$ ,*

$$(1.5) \quad \max_{B_{r(1+1/\sqrt{\lambda})}(x)} |u| \leq c_{10} \max_{B_r(x)} |u|.$$

**Corollary 1.5.** *Under the same condition as above, for  $r \leq c_{11}$ ,*

$$(1.6) \quad \max_{B_r(x)} |\nabla u| \leq \frac{c_{13} \lambda^{3/4}}{r} \max_{B_r(x)} |u|.$$

The idea of this work is derived from our note [4], in which we proved that the Bernstein inequality for a harmonic function is a consequence of the frequency estimate of the function in two dimensions. We would like to thank Prof. S.-T. Yau, J. Spruck, F.-H. Lin and D. Jerison for helpful discussions.

## 2. Bernstein inequality

Let  $(M, g)$  be a connected, smooth, closed two dimensional Riemannian manifold. Suppose that  $\Delta$  is the Laplace-Beltrami operator on  $(M, g)$ , and  $u$  is a real eigenfunction with corresponding eigenvalue  $\lambda$ , i. e.,  $\Delta u = -\lambda u$ . We are only interested in large eigenvalues,  $\lambda \geq 1$ . Let  $H$  be an upper bound of the absolute value of the sectional curvature, and  $D$  be the diameter of the manifold.

Denote  $c$  to be any constant depends only upon  $H$  and  $D$ . Fix the center and we use  $B_r$  to denote a geodesic ball of radius  $r$ . Pulling back the metric to the tangent space via the exponential map. The injective

radius of the new metric is now controlled by  $H$ . Lift the eigenfunction equation and the eigenfunction (by abusing the notation, we still denote it by  $u$ ) to the tangent space and restrict ourself to a smaller convex geodesic ball, we can assume that under polar coordinates  $ds^2 = dr^2 + \rho^2 d\theta^2$ ,

$$\Delta = \frac{1}{\rho} \frac{\partial}{\partial r} \left( \rho \frac{\partial}{\partial r} \right) + \frac{1}{\rho} \frac{\partial}{\partial \theta} \left( \frac{1}{\rho} \frac{\partial}{\partial \theta} \right).$$

Make the radius even smaller, we can further assume that  $\rho$  is a monotonely increasing function in  $r$ . Let  $ds_H^2 = dr^2 + \rho_0^2(r) d\theta^2$  be the canonical metric on the space form with sectional curvature  $-H$ , where  $\rho_0(r) = \sinh \sqrt{H}r / \sqrt{H}$ .

Define  $q = |\nabla u|^2 + \lambda u^2 / 2$  and let  $M(r) = \max_{B_r} q$ .

**Lemma 2.1.**

$$(2.1) \quad \frac{1}{\rho_0} \frac{d}{dr} \left( \rho_0 \frac{d}{dr} \ln M(r) \right) \geq -c_{14} \lambda.$$

*Proof.* Observe that  $M(r)$  is a monotonely non-decreasing function in  $r$  and that  $M(r) \geq q(r, \theta)$ .

Suppose first that  $M(r) > \max_{\partial B_r} q$ .  $M(r)$  is constant near  $r$  and therefore (2.1) holds.

Now suppose that  $M(r) = q(r, \theta_0)$  for certain  $\theta_0$ .  $\ln M - \ln q$  archives its local minimal at  $(r, \theta_0)$ . Hence,  $\Delta \ln M - \Delta \ln q \geq 0$ .

On the other hand, it follows from our results in [6] that

$$\Delta \ln q \geq -\lambda + 2 \min(K, 0) + 4\pi \sum_i (k_i - 1) \delta_{p_i}.$$

Here  $\delta_{p_i}$  is the Dirac function centered at  $p_i$ . This gives us

$$\Delta \ln M(r) \geq -c_{15} \lambda.$$

Finally, the fact that  $\ln M(r)$  is non-decreasing and the comparison theorem for Laplace–Beltrami operators shows that

$$\Delta \ln M(r) \leq \Delta_H \ln M(r) = \frac{1}{\rho_0} \frac{d}{dr} \left( \rho_0 \frac{d}{dr} \ln M(r) \right),$$

where  $\Delta_H$  is the Laplace–Beltrami operator in the space form  $ds_H^2$ .

**Lemma 2.2.** For  $r \leq c_{16} \lambda^{-1/4}$  and  $\epsilon > 0$ , we have

$$(2.2) \quad \frac{1}{\epsilon} (\ln M(r) - \ln M(r(1 - \epsilon))) \leq c_{17} \sqrt{\lambda}.$$

*Proof.* Define

$$t(r) = \int^r \frac{d\tau}{\rho_0(\tau)}.$$

(2.1) can be transformed into

$$\frac{d^2}{dt^2} M \geq -c_{18} \lambda \rho_0^2.$$

Let  $t_1 = t(r(1 - \epsilon))$ ,  $t_2 = t(r)$  and  $t_3 = t(2r)$ . We have  $t_2 - t_1 \leq c_{19} \epsilon$  and  $t_3 - t_2 \geq c_{20}$ . Elementary calculus shows that

$$\begin{aligned} & \frac{\ln M(2r) - \ln M(r)}{t_3 - t_2} - \frac{\ln M(r) - \ln M(r(1 - \epsilon))}{t_2 - t_1} \\ & \geq -c_{21} \lambda \rho_0^2(2r)(t_3 - t_1). \end{aligned}$$

We proved in [6] that

$$\max_{B_{2r}} q \leq e^{c_{22} \sqrt{\lambda}} \max_{B_r} q,$$

or

$$\ln M(2r) - \ln M(r) \leq c_{23} \sqrt{\lambda}.$$

Therefore,

$$\frac{1}{\epsilon} [\ln M(r) - \ln M(r(1 - \epsilon))] \leq c_{24} \sqrt{\lambda} + c_{25} \lambda r^2.$$

Noticing that  $r \leq c_{26} \lambda^{-1/4}$ , we are done.

**Theorem 2.3.** *Let  $M$  be a smooth closed Riemannian manifold of dimension two, and  $u$  be an eigenfunction with eigenvalue  $\lambda$ . Then, for  $r \leq c_{27} \lambda^{-1/4}$*

$$(2.3) \quad \max_{B_r} |\nabla u| \leq \frac{c_{28} \sqrt{\lambda}}{r} \max_{B_r} |u|.$$

*Proof.* Take  $\epsilon = c_{29} / \sqrt{\lambda}$  in Proposition 2.2. Then (2.2) shows that

$$M(r) \leq c_{30} M(r_1),$$

where  $r_1 = r - \delta$  and  $\delta = c_{31}r/\sqrt{\lambda}$ . Suppose  $q(x_0) = M(r_1)$  for some  $x_0 \in \overline{B_{r_1}}$ . Then we have  $\partial B_\delta(x_0) \subset B_r$ , standard elliptic estimate implies that

$$|\nabla u(x_0)| \leq \frac{c_{32}}{\delta^2} \int_{\partial B_\delta(x_0)} |u| \leq \frac{c_{33}}{\delta} \max_{B_r} |u| = \frac{c_{34}\sqrt{\lambda}}{r} \max_{B_r} |u|.$$

Thus,

$$M(r_1) = q(x_0) = |\nabla u(x_0)|^2 + \lambda/2 |u(x_0)|^2 \leq c_{35} \frac{\lambda}{r^2} \left( \max_{B_r} |u| \right)^2.$$

Finally,

$$\max_{B_r} |\nabla u| \leq \sqrt{M(r)} \leq \frac{c_{36}\sqrt{\lambda}}{r} \max_{B_r} |u|.$$

As an application of the theorem, we have

**Corollary 2.4.** *Under the same condition as above, for  $r \leq c_{37}\lambda^{-1/4}$ ,*

$$\max_{B_{r(1+1/\sqrt{\lambda})}} |u| \leq c_{38} \max_{B_r} |u|.$$

*Proof.* Defining  $m(r) = \max_{B_r} |u|$ , the above theorem tells us that

$$\frac{d}{d\tau} m(\tau) \leq \max_{B_r} |\nabla u| \leq \frac{c_{39}\sqrt{\lambda}}{\tau} m(\tau).$$

Therefore,

$$\begin{aligned} \ln m(r(1+\epsilon)) - \ln m(r) &= \int_r^{r(1+\epsilon)} \frac{m'(\tau)}{m(\tau)} d\tau \leq \int_r^{r(1+\epsilon)} \frac{c_{40}\sqrt{\lambda}}{\tau} d\tau \\ &= c_{41}\sqrt{\lambda} \ln(1+\epsilon). \end{aligned}$$

It is obvious that

$$\begin{aligned} \max_{B_{r(1+1/\sqrt{\lambda})}} |u| &= \max \left( \max_{[r, r(1+1/\sqrt{\lambda})]} m(\tau), \max_{B_r} |u| \right) \\ &\leq \max \left( (1+1/\sqrt{\lambda})^{c_{42}\sqrt{\lambda}} m(r), \max_{B_r} |u| \right) \\ &\leq c_{43} \max_{B_r} |u|. \end{aligned}$$

Removing the restriction  $r \leq c_{44}\lambda^{-1/4}$  in Theorem 2.3, we have

**Corollary 2.5.** *Under the same condition as above, for  $r \leq c_{45}$ ,*

$$\max_{B_r} |\nabla u| \leq \frac{c_{46}\lambda^{3/4}}{r} \max_{B_r} |u|.$$

*Proof.* There is nothing more to prove if  $r \leq c_{47}\lambda^{-1/4}$ . Assuming that  $r > c_{48}\lambda^{-1/4}$ , and that  $\max_{B_r} |\nabla u| = |\nabla u(x_0)|$ , for some  $x_0 \in \overline{B_r}$ . Notice that we have taken  $r$  to be sufficiently small so that  $B_r$  is convex. Therefore, we can find a smaller ball  $B_{r'}(p)$  such that  $r' = c_{49}\lambda^{-1/4}$  and  $x \in \overline{B_{r'}(p)} \subset \overline{B_r}$ . Applying the above theorem to  $B_{r'}(p)$  and observing that  $\max_{B_{r'}(p)} |u| \leq \max_{B_r} |u|$ , we have

$$\max_{B_r} |\nabla u| = \max_{B_{r'}(p)} |\nabla u| \leq \frac{c_{50}\lambda^{1/2}}{r'} \max_{B_r} |u| \leq \frac{c_{51}\lambda^{3/4}}{r} \max_{B_r} |u|.$$

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